The Edge-to-vertex Geodetic Number of some snake Graphs

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Abstract

A set $S \subseteq E$ is called an edge-to-vertex geodetic set of G if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The minimum cardinality of an edge-to-vertex geodetic set of G is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an edge-to-vertex geodetic basis of G. In this paper we study the edge-to-vertex geodetic number of some path related graphs called snake graphs which are obtained from the path P_n by replacing its edges by cycles C_3.

Keywords: geodesic, edge-to-vertex geodetic set, edge-to-vertex geodetic number.

AMS Subject Classification: 05C12.

1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1, 5]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. The *geodetic number* $g(G)$ of G is the minimum order of a geodetic set and any geodetic set of order $g(G)$ is called a *geodetic basis* of G. The geodetic number of a graph was studied in [1, 2, 3,4]. For subsets A and B of $V(G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d(A, B)$ is called an $A - B$ *geodesic* joining the sets A, B where $u \in A$ and $v \in B$. A vertex x is said to *lie* on an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A - B$ geodesic as $uv - zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G. The edge-to-vertex geodetic number of a graph was introduced by Santhakumaran and John and the same was further studied by various authors in [6]. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. A vertex v is an *end vertex* of a graph G if $d(v)=1$. A *cut-vertex* (*cut-edge*) of a graph G is a vertex (edge) whose removal increases the number of components. Two vertices u and v of G are *antipodal* if $d(u, v) = diam\ G$ or $d(G)$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n. The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3. The double triangular snake DT_n consists of two triangular snakes that have a common path. The alternate triangular snake AT_n is obtained from a path P_n by replacing every alternate edge of a path P_n by a cycle C_3. The double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes which have a common path. The quadrilateral snake Q_n is obtained from a path P_n by replacing every edge of a path P_n by a cycle C_4.Throughout this paper G denotes a connected graph with at least three vertices. The following theorems are used in sequel.

Theorem 1.1. [6] If v is an extreme vertex of a connected graph G, then every edge-to-vertex geodetic set contains at least one extreme edge that is incident with v.

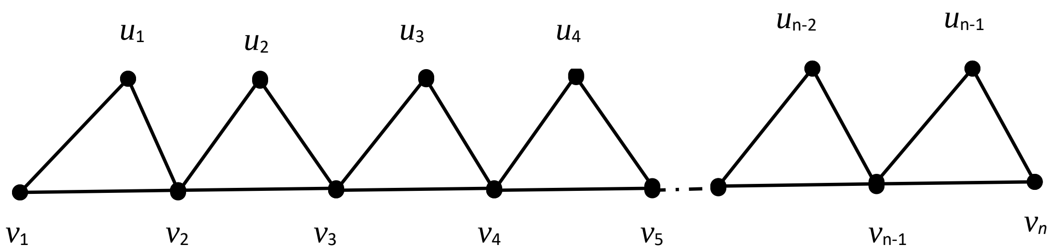
Theorem 1.2. [6] Let G be a connected graph and S be a g_{ev}-set of G. Then no cut edge of G which is not an end-edge of G belongs to S.

Theorem 1.3. [6] Every end-edge of a connected graph G belongs to every edge-to-vertex geodetic set of G.

2. Main Results

Theorem 2.1. For the triangular snake $G= T_n$, $g_{ev}(G) = n$-1.

Proof. Consider the path $P_n : v_1, v_2, v_3, v_4\ldots, v_{n-1}, v_n$. Let the triangular snake T_n in Figure 2.1 be obtained by replacing each edge v_iv_{i+1} of P_n to triangle C_3 by adding the new vertices $u_1, u_2, u_3, u_4\ldots, u_{n-1}$. The triangular snake T_n consists of $2n$-1 vertices, 3(n-1) edges and n-1 triangles. Moreover, it consists of $2n$ extreme edges. (Each C_{i},,i=2,3… n-2 has two extreme edges and C_1 and C_n have three extreme edges) By Theorem 1.1, every edge-to-vertex geodetic set contains at least one extreme edge from each C_3, we have $g_{ev}(G) \geq n$-1. Suppose that $g_{ev}(G) = n$. Then there exists a mínimum edge-to-vertex geodetic set S such that $|S| = n$. Without loss of generality, let us take $S = \{u_1v_1, u_2v_2, u_3v_3, \ldots, u_{n-1}v_{n-1}, u_{n-1}v_n\}$. Clearly S is an edge-to-vertex geodetic set of G. But S-$\{u_{n-1}v_{n-1}\}$ is an edge-to-vertex geodetic set of G and is contained in S. So S is not a minimum edge-to-vertex geodetic set. Therefore, $g_{ev}(G) \leq n$-1.Hence $g_{ev}(G) = n$-1.

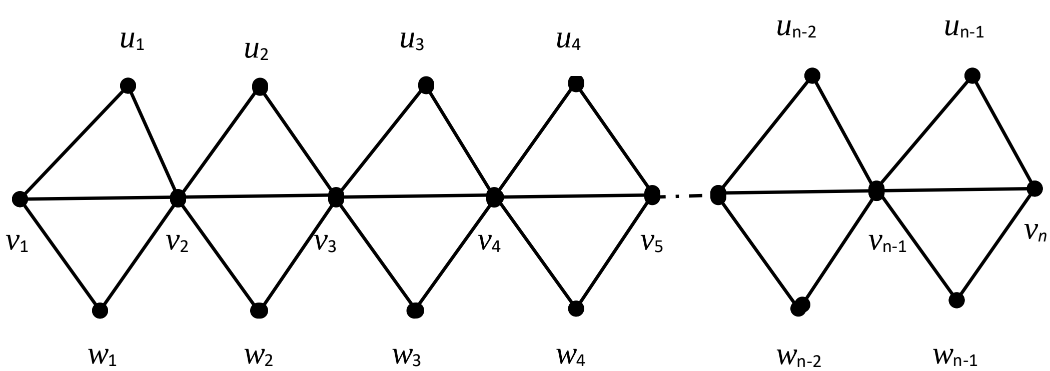


Triangular snake T_n

Figure 2.1

Theorem 2.2. For the double triangular snake $G = DT_n$, $g_{ev}(G) = 2(n$-1).

Proof. Consider the path P_n : $v_1, v_2, v_3, v_4 \ldots, v_{n-1}, v_n$. The doublé triangular snake DT_n in Figure 2.2 is obtained by replacing each edge $v_i\, v_{i+1}$ of P_n to two triangle's C_3 in which the path is common for both the triangles and the new vertices are $u_1, u_2, u_3, u_4 \ldots, u_{n-1}$ and $w_1, w_2, w_3, w_4 \ldots, w_{n-1}$. The doublé triangular snake consists of $3n$-2 vertices, $5(n$-1) edges and $2(n$-1) triangles. Clearly DT_n has 4(n-1) extreme edges. By Theorem 1.1, every edge-to-vertex geodetic set contains at least one extreme edge from each C_3, we have $g_{ev}(G) \geq 2(n$-1). Let $S = \{u_1v_1, v_1w_1, u_2v_3, v_3w_2, u_3v_4, v_4w_3, \ldots, u_{n-1}v_n, v_nw_{n-1}\}$ be a subset of the set of all extreme edges of G. It is easily observe that S is a minimum edge-to-vertex geodetic set of G, and $|S| = 2(n-1)$. Therefore, $g_{ev}(G) \leq 2(n$-1). Hence $g_{ev}(G) = 2(n$-1).



Double Triangular snake DT_n

Figure 2.2

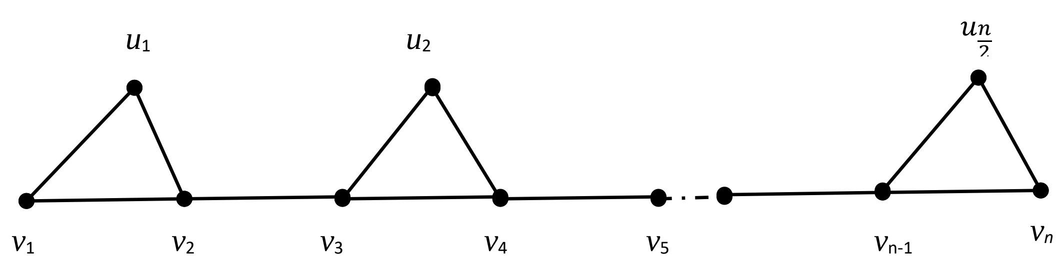
Remark 2.3. For the above two theorems, we can see that the edge-to-vertex geodetic number of T_n and DT_n depends on the number of triangles in the corresponding snake graph.

Theorem 2.4. For an alternate triangular snake $G = AT_n$,

$$g_{ev}(G) = \begin{cases} \frac{n}{2} \ \textit{if the path } P_n \textit{ is even} \\ \left\lceil \frac{n}{2} \right\rceil \textit{if the path } P_n \textit{ is odd} \end{cases}$$

Proof. Case (i) n is even and $n \geq 4$.

Consider the path P_n : $v_1, v_2, v_3, v_4 \ldots, v_{n-1}, v_n$ where n is even. The alternate triangular snake AT_n, in Figure 2.3 is obtained by replacing the alternate edges of P_n by triangle C_3. Clearly AT_n contains $\frac{n}{2}$ triangles in which $u_1, u_2, u_3, u_4 \ldots, u_{n/2}$ are the new vértices. Note that AT_n has n extreme edges and $\frac{n}{2} - 1$ cut edges. By Theorem 1.1, every edge-to-vertex geodetic set contains at least one extreme edge from each C_3, and hence $g_{ev}(G) \geq \frac{n}{2}$. Also by Theorem 1.2, no cut edge of G which is not an end-edge of G belongs to every edge-to-vertex geodetic set of G. Let $S = \{u_1v_1, u_2v_4, u_3v_6, \ldots, u_{\frac{n}{2}}v_n\}$. Clearly S is a subset of the set of all extreme edges of $G = AT_n$. Since every vertices of AT_n are either in S or lies in a geodesic joining of some pair of edges of AT_n, we get S is an edge-to-vertex geodetic set of $G = AT_n$. Also it is seen that S is a minimum edge-to-vertex geodetic set of AT_n. Therefore $g_{ev}(G) = |S| = \frac{n}{2}$.

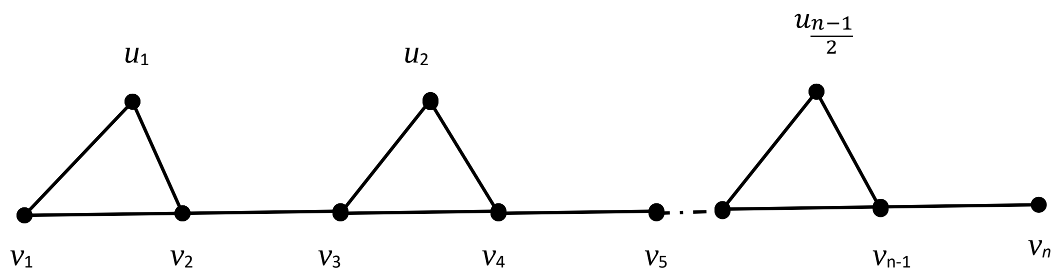


Alternate Triangular snake AT_n

Figure 2.3

Case (ii) n is odd and $n \geq 3$.

In this case the alternate triangular snake AT_n in Figure 2.4 contains an end edge, $\frac{n-1}{2}$ triangles and $\frac{n-3}{2}$ cut edges. It is easily observe that AT_n has n extreme edges. By Theorem 1.3 & 1.1, Every edge-to-vertex geodetic set S of AT_n contains an end edge and at least $\frac{n-1}{2}$ extreme edges and hence $g_{ev}(G) \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Consider the set $S = \{u_1v_1, u_2v_4, u_3v_6, \ldots, u_{\frac{n-1}{2}} v_{n-1}, v_{n-1}v_n\}$. Clearly S is a minimum edge-to-vertex geodetic set of AT_n. Hence $g_{ev}(G) = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.



Alternate triangular snake AT_n

Figure 2.4

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